Acoustic Einstein–Hopf drag on a bubble†

A. Larraza and E. Tucholski
Department of Physics – Code PH/La
Naval Postgraduate School
Monterey, CA 93943

Theoretical results show that the drag on a bubble can be modified by the presence of isotropic, homogeneous, broadband acoustic noise, when the band overlaps the bubble's resonance width. While these results constitute an acoustic analog to the Einstein-Hopf drag on an oscillating dipole in the presence of electromagnetic fluctuations, an important difference is that band-limited acoustic noise can *reduce* the drag when the lower frequency of the spectrum coincides with the resonant frequency of the bubble. Applications to bubble migration, heat transfer, and acoustophoresis are suggested.

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The notion that acoustic noise can test, by analogy, predictions due to stochastic electrodynamics and to electromagnetic zero-point-field (ZPF) effects has been established recently by measurements of the force law between two rigid, parallel plates due to the radiation pressure of broadband acoustic noise [1]. This measurement constitutes an acoustic analog to the Casimir effect [2], which is the force between two closely spaced uncharged parallel conducting plates due to the radiation pressure of the ZPF. In this work, we report on theory that shows that the drag on a bubble can be modified by the presence of isotropic, homogeneous, broadband acoustic noise, when the band overlaps the bubble's resonance width. This acoustic–induced drag is the acoustic analog to the Einstein–Hopf drag [3].

Einstein and Hopf studied a simple model for the thermal equilibrium between oscillating dipoles and isotropic, homogeneous, electromagnetic thermal fluctuations. Due to recoil associated with emission and absorption of fluctuating electromagnetic radiation, a particle experiences a random walk in phase space, leading to an average *growth* in its kinetic energy. However, this accelerating effect is balanced by a dissipative, velocity dependent drag force, because of Doppler shifts. Einstein and Hopf appreciated the energy balancing action of the two opposing effects, which we understand today as a manifestation of the fluctuation–dissipation theorem. They showed that this energy balance, and the application of the equipartition theorem for energy solely to the translational motion of the oscillator, leads to the Rayleigh radiation law.

In the model considered by Einstein and Hopf, a dipole is composed of a mass m and charge e bounded by an elastic restoring force to a mass M>>m of opposite charge. Einstein and Hopf restricted the oscillations of the dipole to one direction, for which the equation of motion is

$$\frac{d^2p}{dt^2} - \Gamma\frac{d^3p}{dt^3} + \omega_0^2 p = \frac{3}{2}\Gamma c^3 E_z \quad , \tag{1}$$

where p is the oscillator dipole moment, $\Gamma = 2e^2/3mc^3$ is the radiation damping constant, c is the speed of light, and ω_0 is the characteristic frequency of the oscillator. In Eq. (1) we have assumed a dipole oriented along the z–direction, with E_z the z–component of the electric field of the random radiation.

When viewed from a moving particle, the fields experienced by the particle are Lorentz–transformed to a frame moving with the particle. In this frame, the spectrum of radiation loses its isotropy, giving rise to a velocity–dependent force. If we assume translational motion along the x–axis, the force on the particle due to the interaction of the dipole with the electromagnetic fluctuations is

$$F_x{}' = \frac{\partial E_x{}'}{\partial z'} p' - B_y{}' \frac{dp'}{dt'} \quad , \tag{2}$$

where primed quantities are evaluated in the particle's frame. Evaluating Eq. (2) for an electromagnetic thermal spectrum $\varepsilon(\omega,T)$ leads to the velocity dependent drag force [3, 4]

$$F_x{}' = -\frac{6}{5}\pi^2 \Gamma c\left(\varepsilon(\omega_0,T) - \frac{1}{3}\omega_0 \frac{\partial \varepsilon(\omega_0,T)}{\partial \omega_0}\right) v \quad , \tag{3}$$

where v is the velocity of the particle, and the spectrum is evaluated at the characteristic frequency of the dipole. In thermodynamic equilibrium the expression in brackets is non–negative, and vanishes for the special case of a spectrum proportional to the cube of the

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frequency. This corresponds to the zero−temperature limit value of $\varepsilon(\omega,T=0)$, or zero−point field spectrum, which has the form $\varepsilon_0(\omega) = \hbar\omega^3/2\pi^2c^3$, where $\hbar$ is the reduced Planck's constant.

In the acoustic analog, electromagnetic thermal fluctuations are replaced by an externally−imposed isotropic and homogeneous acoustic noise. In a fluid of density ρ and speed of sound c_s, the fluctuating acoustic pressure field can be written as traveling waves of the form

$$p_a(\mathbf{x},t) = \rho c_s \int d^3k f(\omega)\cos(\omega t - \mathbf{k}\cdot\mathbf{x} - \theta_\mathbf{k}), \qquad (4)$$

where $\omega = c_s k$. Here the random phase $\theta_\mathbf{k}$ has been introduced to indicate the fluctuating character of the acoustic noise. For isotropic noise, the function $f(\omega)$ can only depend on ω, and it is connected to the spectral energy density $\varepsilon(\omega)$ by

$$\varepsilon(\omega) = \frac{2\pi\rho}{c_s^3}\omega^2 f^2(\omega)\cdot \qquad (5)$$

In a frame moving with velocity $\mathbf{v}$, the acoustic noise spectrum is no longer isotropic, and in this frame, the pressure can be written as

$$p_a^/(\mathbf{x}^/,t) = \rho c_s \int d^3k f(\omega^/)\cos(\omega^/ t - \mathbf{k}\cdot\mathbf{x}^/ - \theta_\mathbf{k}), \qquad (6)$$

where $\omega^/ = \omega + \mathbf{v}\cdot\mathbf{k}$ and $\mathbf{k}^/ = \mathbf{k}$.

Consider now the volume oscillations of a bubble induced by the acoustic field. For small volume oscillations, the equation of motion for the volume $V^/$ in the instantaneous frame of the bubble is

$$\ddot{V}^/ + R\dot{V}^/ + \omega_B^2(V^/ - V_0) = \frac{S_0^2}{m}(p_0 - p_e^/), \qquad (7)$$

where p_0 is the hydrostatic pressure at which the bubble has the mean volume V_0, S_0 is the equilibrium surface area of the bubble, $p_e^/ = p_0 + p_a^/$ is the instantaneous external pressure that would exist in the liquid at the bubble location in the absence of the bubble, R is a measure of the damping effects (thermal, radiation, and viscous), ω_B is the characteristic frequency of oscillations of the bubble, and $m \approx 3\rho V_0$ is the entrained mass of the fluid, which is about three times the mass of the fluid displaced. Equation (7) applies for wavelengths of sound much larger than the bubble radius. From Eqs. (6) and (7) we can solve for the volume oscillations $\delta V^/ = V^/ - V_0$

$$\delta V^/ = -\frac{\rho c_s S_0^2}{mR}\int d^3k \frac{f(\omega^/)}{\omega^/}\sin\alpha(\omega^/)\cos(\omega^/ t - \alpha - \theta_\mathbf{k}), \qquad (8)$$

where $\cot\alpha(\omega^/) = (\omega_B{}^2 - \omega^{/2})/R\omega^/$. Because the origin of the moving frame coincides instantaneously with the bubble at time t, the phase term $-\mathbf{k}\cdot\mathbf{x}^/$ is absent from the argument of the cosine.

For wavelengths of sound much greater than the size of the bubble, the translational force exerted on the bubble by the sound field is equal to the bubble volume times the negative gradient of the acoustic pressure at the bubble's location. The average acoustic force is then given by the ensemble average over random phases

$$\mathbf{F} = -\left\langle \delta V^/ \nabla p_a^/ \right\rangle = \frac{\rho^2 c_s{}^2 S_0^2}{2mR}\int d^3k \frac{f^2(\omega^/)}{\omega^/}\sin^2\alpha(\omega^/)\mathbf{k}\cdot \qquad (9)$$

For velocities $v \ll c_s$, and small damping, $R \ll \omega_B$, Eq. (9) leads to the velocity−dependent drag force

$$\mathbf{F} = -\pi^2 a\left(\varepsilon(\omega_B) - \frac{1}{3}\omega_B\frac{\partial\varepsilon(\omega_B)}{\partial\omega_B}\right)\mathbf{v}, \qquad (10)$$

where *a* is the bubble radius at the equilibrium volume V_0.

In the acoustic analog to the Casimir effect [1], band limited acoustic noise yields an effect not considered in the electromagnetic counterpart, namely it can cause the force to be attractive *or* repulsive as a function of separation between the plates. In the present analog, band-limited acoustic noise also causes effects not considered in the electromagnetic Einstein-Hopf. For some spectral shapes, if the lower frequency of the spectrum coincides with the resonant frequency of the bubble, the force (10) can act in the direction of motion. That is, the noise exerts a *negative* drag on the bubble in this case. Thus, besides providing the acoustic analog to the Einstein−Hopf drag, our investigations can lead to analogs to mechanisms for stochastic acceleration of charged particles that are used to explain cosmic rays.

The drag force (10) is in addition to the hydrodynamic drag force on a bubble. For millimeter−size air bubbles in water, the typical Reynolds number is about 60, and we may use the analytical expression derived by Moore [5] to determine the hydrodynamic drag. Moore's expression for the drag coefficient allows for interfacial slippage and it is given by

$$C_D = \frac{48}{R_e}\left(1 - \frac{2.2}{R_e{}^{1/2}}\right), \qquad (11)$$

where $R_e=2av/\nu$ is the Reynolds number for a bubble of radius *a* moving with velocity v in a fluid with kinematic viscosity ν.

The resonance frequency of volume oscillations of air bubbles is determined by the compressibility of the internal gas and the entrained inertia of the liquid. For bubbles that are greater than 10 μm in diameter, the resonance frequency (in Hz) of a bubble in water at atmospheric pressure can be determined by dividing 3.28 by the bubble radius in meters. Bubbles with resonance frequencies of about 10 kHz (~0.3 mm) have typical Q values of about 15, corresponding to a bandwidth of about 0.7 kHz.

To determine the order of magnitude effect of the acoustic–induced drag (10), consider a flat noise spectral distribution $\varepsilon(\omega)$ with a roll–off of 114 dB/octave [6] and with an rms acoustic pressure of 40 kPa in a band of frequencies between 8–10 kHz (Figure 1). Figure 2 shows a plot of the terminal velocity as a function of bubble resonance frequency when the buoyant force of an ascending bubble, the hydrodynamic drag force, and the acoustic–induced drag force balance each other. The terminal velocity is normalized to its value in the absence of acoustic noise. When the bubble's resonance width does not overlap with the band of frequencies of the noise, the drag on the bubble is not modified by the presence of the acoustic noise. However, if the resonant frequency of the bubble coincides with the upper frequency of the spectrum of Figure 1, the terminal velocity is reduced by 20% of its sound field–free value, and it is reduced by 2% if the bubble's resonance width overlaps with the band of frequencies of the noise. When the resonant frequency of the bubble coincides with the lower frequency of the spectrum of Figure 1, the noise exerts a *negative* drag on the bubble, and the terminal velocity is bigger than its sound field–free value by 30%. In an experiment in progress, we are seeking to measure the large effects predicted to occur for bubbles with resonant frequencies equal or near the edges of a spectrum like the one in Figure 1.

The modifications of the drag experienced by a bubble in the presence of acoustic noise suggest possible applications to bubble migration and to heat transfer in a two-phase fluid, and also present the attractive possibility of controlling the reaction rates in chemical engineering processes carried out in reaction beds in a largely bubbly environment.

The acoustic drag experienced by particles undergoing volume oscillations also suggests possible applications to acoustophoresis, separation of particles using high intensity sound waves. Solid particles similar in size, charge, and density cannot be separated by filtration, electrophoresis, or centrifugation. However if the particles have different elastic properties, their resonant frequencies are different. As shown in Figure 2, a high intensity band–limited noise spectrum with a sharp roll–off acts as a discrete separator of these particles while they are carried by an external flow.

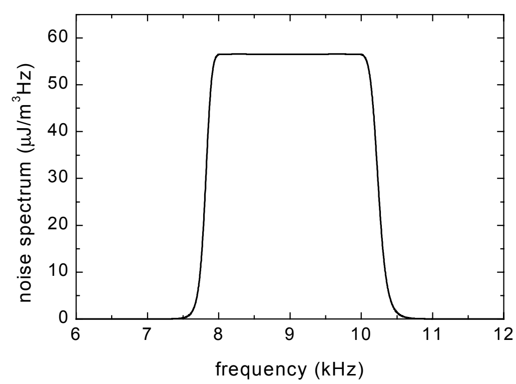

Figure 1 Acoustic noise spectrum in a band of frequencies between 8–10 kHz, with an rms pressure of 40 kPa. The spectrum corresponds to the transfer function of a 19–pole (114 dB/octave roll–off) Tchebyshev bandpass filter.

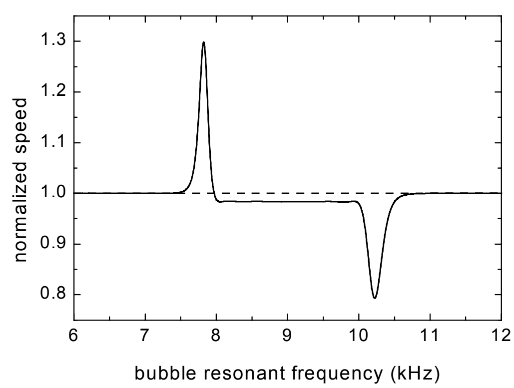

Figure 2 Terminal velocity as a function of bubble resonance frequency. The velocity is normalized to its value in the absence of acoustic noise.

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[6] The Stanford Research System™ Model 650 is an example of a commercially available 115 dB/octave bandpass filter.